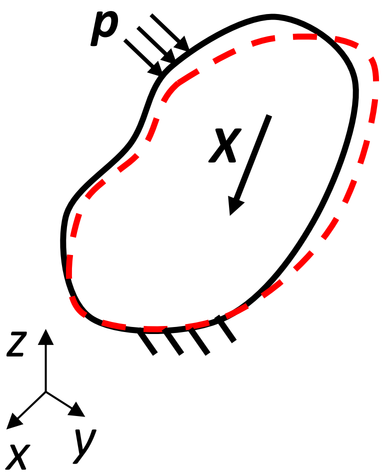
Finite element method (FEM1)

Lecture 1. Approximate numerical methods

02.2026

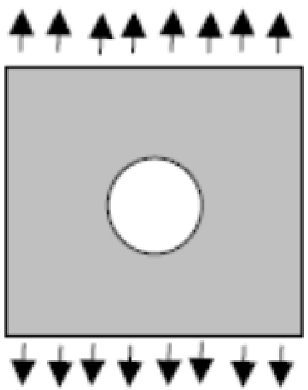
Approximate methods in the analysis of continuous media

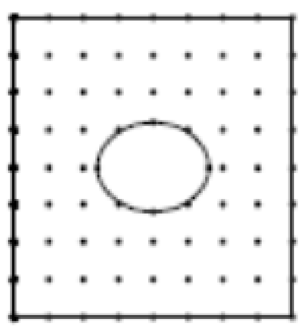


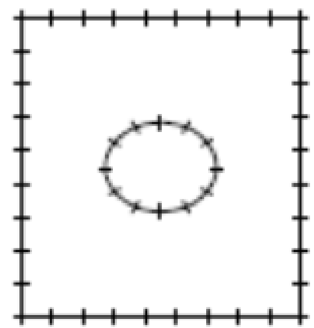
Deformable structures can be examined:

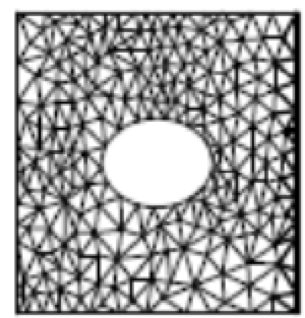
- experimental methods *(cost, time consuming)*
- theoretical methods:
  - analytical *(only simple models)*
  - numerical *(approximate methods) – FDM, BEM, FEM*

Finite difference method (FDM)	*Boundary element method (BEM)*	*Finite element method (FEM)*
Partial differential equations	Boundary integral equations	Functional minimalization





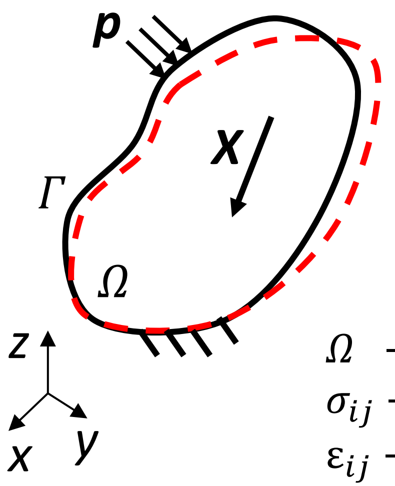




In approximate methods, the problem of searching for unknown functions (e.g. describing the displacement field) is replaced by the problem of searching for a finite number of parameters.

Principle of minimum total potential energy

The Finite Element Method in structural statics is usually presented as an approximate method using the theorem of the minimum total potential energy of a deformable system.



Total potential energy of a deformable system:

$$V = U - W = \frac{1}{2}\int_{\Omega} \sigma_{ij}\varepsilon_{ij}d\Omega - \int_{\Omega} X_i \mathrm{u}_i d\Omega - \int_{\Gamma} p_i \mathrm{u}_i d\Gamma$$

Ω – *domain* , Γ – *boundary* , u_i – *displacement vector*,
σ_{ij} – *stress tensor*, p_i – *surface load*,
ε_{ij} – *strain tensor*, X_i – *mass forces*

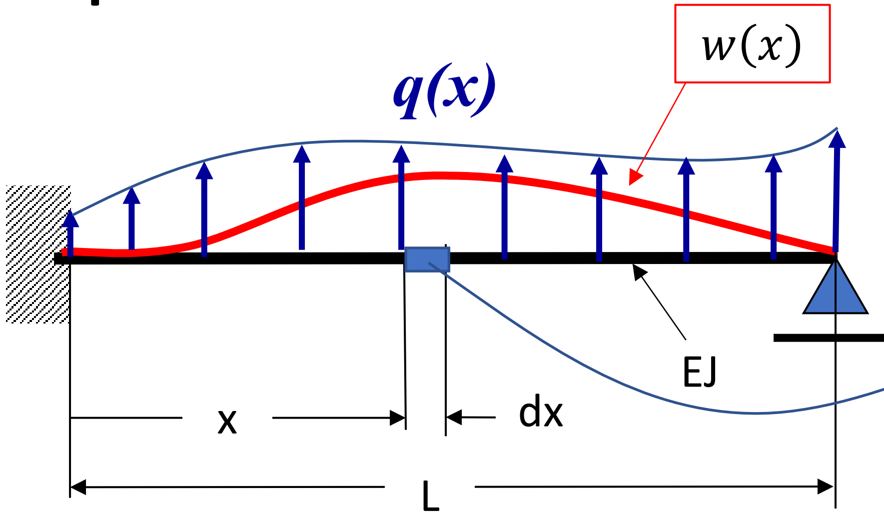
The principle of minimum total potential energy states that:
Of all geometrically admissible forms of displacements to which an elastic system may be subjected, the one for which the functional of the total potential energy reaches a minimum value will occur.

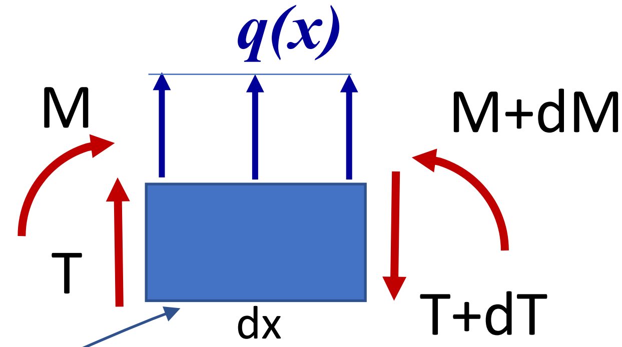
$$V = U - W = min\ !$$

V *– functional (for the searched function $u(\bar{x})$ we have a numerical value)*

Minimizing the functional is the task of the calculus of variations

Example 1: Deflection line of a beam





$$q = \frac{dT}{dx} \qquad T = \frac{dM}{dx}$$

Boundary conditions: $w(x=0)=0$ $\quad w(x=L)=0$ $\quad \frac{dw}{dx}(x=0)=0$

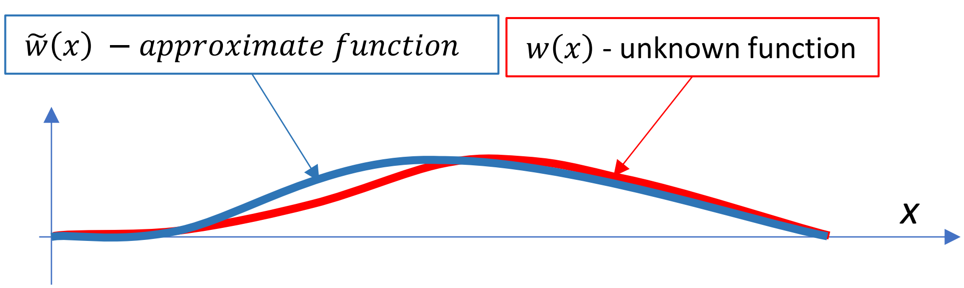
Differential equation

$$\frac{d^2}{dx^2}\left(EJ\,\frac{d^2w}{dx^2}\right) = q(x\,)$$

or

Principle of minimum total potential energy

$$V(w) = \frac{1}{2}\int_0^L EJ\,(w'')^2 dx - \int_0^L q(x)w(x)\,dx = min$$



Approximate function:

$$\tilde{w}(x) = \sum_{i=1}^{n} a_i\, g_i(x)$$

Approximation: *parametric or nodal*

global or local

The Ritz Method

Let's introduce an approximating function:

$$\widetilde{w}(x) = a_1 \cdot g_1(x) + a_2 \cdot g_2(x) + \ldots + a_n \cdot g_n(x)$$

(it can be a power series or Fourier series)

The approximating function is a linear combination of the unknown parameters $\boldsymbol{a_i}$ and known geometrically feasible functions $\boldsymbol{g_i(x)}$

After substituting this function into the expression for the total potential energy, we obtain the function of parameters $\boldsymbol{a_i}$:

$$V = \frac{1}{2}\int_0^L EJ\,(\widetilde{w}'')^2 dx - \int_0^L q(x)\widetilde{w}(x)\,dx \quad \rightarrow \quad V = f(a_1, a_2, \ldots, a_n)$$

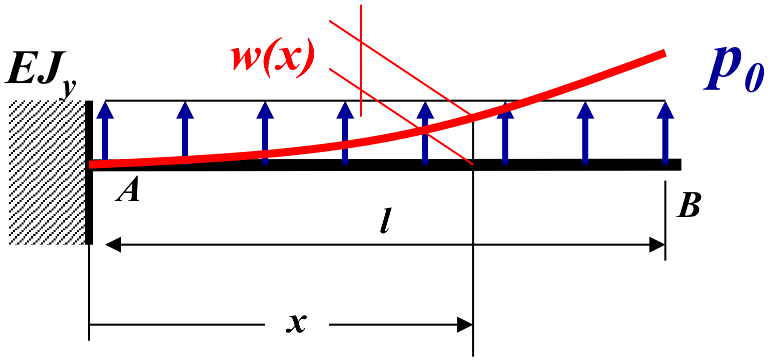
Now we need to find the minimum of the function with respect to the parameters:

$$\frac{\partial V}{\partial a_1} = 0\,; \quad \frac{\partial V}{\partial a_2} = 0 \quad \ldots \quad \frac{\partial V}{\partial a_n} = 0$$

Finally, we get a system of linear algebraic equations

$$\rightarrow \quad a_i \quad \rightarrow \quad \widetilde{w}(x)$$

Example 2: Cantilever beam loaded with uniform distributed forces



Solve a cantilever beam using the given approximating function using the Ritz method:

$$\tilde{w}(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3$$

Polynomial of 3rd order

Boundary conditions: $\tilde{w}(x=0)=0 \rightarrow a_1 = 0$ $\quad \tilde{w}'(x=0)=0 \rightarrow a_2 = 0$

A function satisfying the geometric conditions is given as:

$$\tilde{w}(x) = a_3 \cdot x^2 + a_4 \cdot x^3 \rightarrow \tilde{w}'(x) = 2a_3 \cdot x + 3a_4 \cdot x^2 \rightarrow \tilde{w}''(x) = 2a_3 + 6a_4 \cdot x$$

Potential energy:

$$V = \frac{1}{2}\int_l EJ[w''(x)]^2 dx - \int_0^l p_0(x) \cdot w(x) dx$$

After substituting the approximating function:

$$V = \frac{1}{2}\int_l EJ[\tilde{w}''(x)]^2 dx - \int_0^l p_0(x) \cdot \tilde{w}(x) dx$$

$$V = \frac{EJ}{2}\int_0^l (2a_3 + 6a_4 x)^2 dx - p_0 \int_0^l \left(a_3 x^2 + a_4 x^3\right) dx$$

$$V = \frac{EJ}{2}\int_0^l \left(4a_3^2 + 24 a_3 a_4 x + 36 a_4^2 x^2\right) dx - p_0 \int_0^l \left(a_3 x^2 + a_4 x^3\right) dx$$

Example 2: Cantilever beam (cont.)

$$V=\tfrac{EJ}{2}\int_0^l\left(4a_3^2+24a_3a_4x+36a_4^2x^2\right)dx-p_0\int_0^l\left(a_3x^2+a_4x^3\right)dx$$

$$V=\tfrac{EJ}{2}\left(4a_3^2x+12a_3a_4x^2+12a_4^2x^3\right)\Big|_0^l-p_0\left(\tfrac{1}{3}a_3x^3+\tfrac{1}{4}a_4x^4\right)\Big|_0^l$$

$$V=\tfrac{EJ}{2}\left(4a_3^2l+12a_3a_4l^2+12a_4^2l^3\right)-p_0\left(\tfrac{1}{3}a_3l^3+\tfrac{1}{4}a_4l^4\right)$$

The minimum condition for a function:

$$\begin{cases}\dfrac{\partial V}{\partial a_3}=0 \rightarrow \dfrac{\partial V}{\partial a_3}=\tfrac{EJ}{2}\left(8la_3+12l^2a_4\right)-\tfrac{1}{3}p_0l^3=0\\[2ex] \dfrac{\partial V}{\partial a_4}=0 \rightarrow \dfrac{\partial V}{\partial a_4}=\tfrac{EJ}{2}\left(12l^2a_3+24l^3a_4\right)-\tfrac{1}{4}p_0l^4=0\end{cases}$$

$$\rightarrow \quad a_3=\frac{5}{24}\frac{p_0l^2}{EJ_y} \qquad a_4=-\frac{1}{12}\frac{p_0l}{EJ_y}$$

Final approximating function:

$$\widetilde{w}(x)=\frac{5}{24}\frac{p_0l^2}{EJ_y}\cdot x^2-\frac{1}{12}\frac{p_0l}{EJ_y}\cdot x^3$$

Example 2: Cantilever beam (cont.)

Final approximating function:

$$\widetilde{w}(x) = \frac{5}{24}\frac{p_0 l^2}{EJ_y}\cdot x^2 - \frac{1}{12}\frac{p_0 l}{EJ_y}\cdot x^3$$

Bending moment approximation:

$$\widetilde{M}_B = EJ_y \widetilde{w}''(x) \rightarrow \widetilde{M}_B(x) = \frac{5}{12} p_0 l^2 - \frac{1}{2} p_0 l \cdot x$$

Approximation of shear force:

$$\widetilde{T} = EJ_y \widetilde{w}'''(x) \rightarrow \widetilde{T}(x) = -\frac{1}{2} p_0 l$$

The exact solution:

$$w(x) = \frac{6}{24}\frac{p_0 l^2}{EJ_y}\cdot x^2 - \frac{2}{12}\frac{p_0 l}{EJ_y}\cdot x^3 + \frac{1}{24}\frac{p_0}{EJ_y}\cdot x^4$$

$$M_B(x) = \frac{1}{2} p_0 (l-x)^2$$

$$T(x) = -p_0(l-x)$$

Example 2: Cantilever beam (cont.)

Approximate solution:

$$\tilde{w}(x) = \frac{5}{24}\frac{p_0 l^2}{EJ_y}\cdot x^2 - \frac{1}{12}\frac{p_0 l}{EJ_y}\cdot x^3$$

$$\tilde{M}_B(x) = \frac{5}{12}p_0 l^2 - \frac{1}{2}p_0 l \cdot x$$

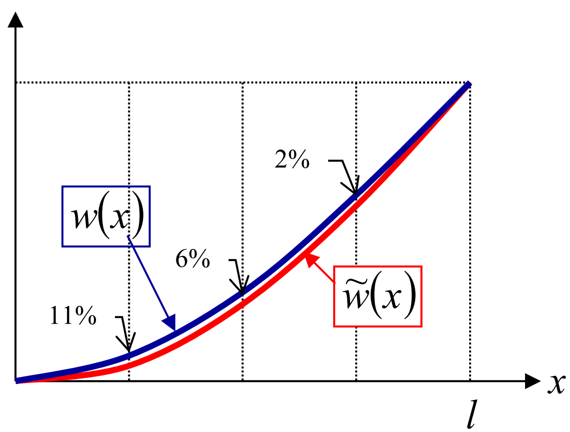
$$\tilde{T}(x) = -\frac{1}{2}p_0 l$$

The exact solution:

$$w(x) = \frac{6}{24}\frac{p_0 l^2}{EJ_y}\cdot x^2 - \frac{2}{12}\frac{p_0 l}{EJ_y}\cdot x^3 + \frac{1}{24}\frac{p_0}{EJ_y}\cdot x^4$$

$$M_B(x) = \frac{1}{2}p_0 (l - x)^2$$

$$T(x) = -p_0 (l - x)$$



$$\tilde{w}(l) = 0.125\frac{p_0 l^4}{EJ_y}$$

$$\tilde{w}\left(\tfrac{3}{4}l\right) = 0.082\frac{p_0 l^4}{EJ_y}$$

$$\tilde{w}\left(\tfrac{1}{2}l\right) = 0.042\frac{p_0 l^4}{EJ_y}$$

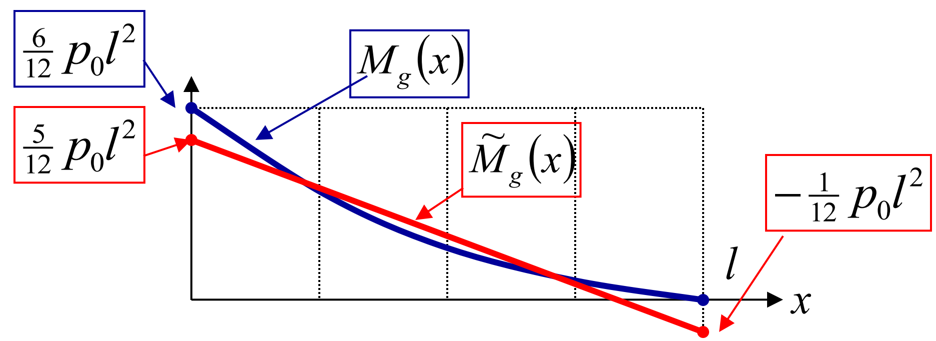
$$\tilde{w}\left(\tfrac{1}{4}l\right) = 0.012\frac{p_0 l^4}{EJ_y}$$

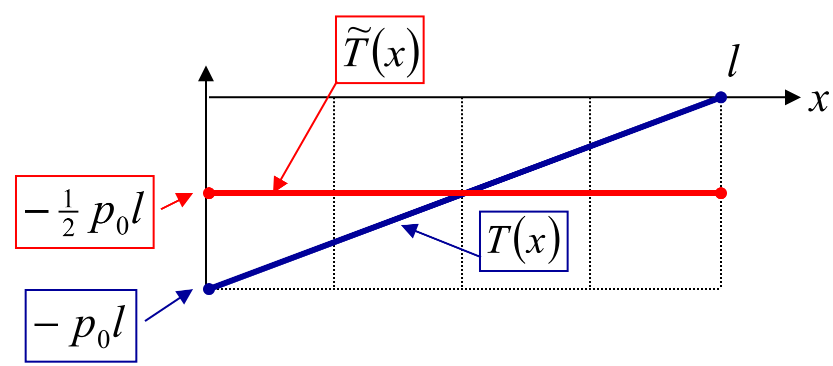
$$w(l) = 0.125\frac{p_0 l^4}{EJ_y}$$

$$w\left(\tfrac{3}{4}l\right) = 0.084\frac{p_0 l^4}{EJ_y}$$

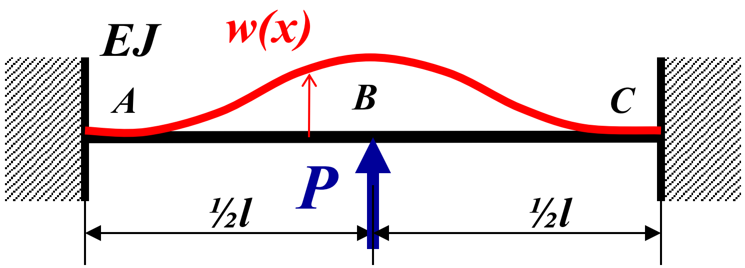
$$w\left(\tfrac{1}{2}l\right) = 0.044\frac{p_0 l^4}{EJ_y}$$

$$w\left(\tfrac{1}{4}l\right) = 0.013\frac{p_0 l^4}{EJ_y}$$





Example 3: Beam fixed on both ends



Solve a statically indeterminate beam using the Ritz method and a given approximating function:

$$\widetilde{w}(x) = A \cdot \left(1 - \cos\frac{2\pi x}{l}\right)$$

$$\widetilde{w}'(x) = \frac{2\pi}{l} A \cdot \sin\frac{2\pi x}{l}$$

$$\widetilde{w}''(x) = \frac{4\pi^2}{l^2} A \cdot \cos\frac{2\pi x}{l}$$

<u>Boundary conditions</u>: $\widetilde{w}(x=0)=0$ $\quad \widetilde{w}'(x=0)=0$ $\quad \widetilde{w}(x=l)=0$ $\quad \widetilde{w}'(x=l)=0$

The function satisfies the geometric conditions!

Total potential energy:

$$V = \frac{1}{2}\int_l EJ\left[w''(x)\right]^2 dx - P \cdot w\left(\tfrac{1}{2} l\right)$$

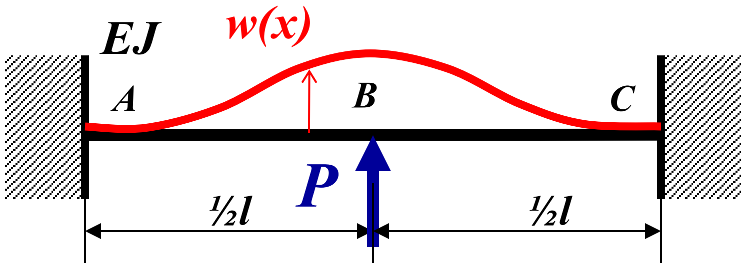
After substituting the approximating function:

$$V = \frac{1}{2}\int_l EJ\left[\widetilde{w}''(x)\right]^2 dx - P \cdot \widetilde{w}\left(\tfrac{1}{2} l\right)$$

$$V = \frac{EJ}{2}\int_0^l \left(\frac{4\pi^2}{l^2} A \cdot \cos\frac{2\pi x}{l}\right)^2 dx - P \cdot A \cdot \left(1 - \cos\frac{2\pi \frac{1}{2} l}{l}\right)$$

$$V = \frac{EJ}{2}\frac{16\pi^4}{l^4} A^2 \cdot \int_0^l \cos^2\frac{2\pi x}{l} dx - 2P \cdot A$$

Example 3: Beam fixed on both ends (cont.)



$$\widetilde{w}(x) = A \cdot \left(1 - \cos\frac{2\pi x}{l}\right)$$

$$V = \frac{EJ}{2}\frac{16\pi^4}{l^4}A^2 \cdot \int_0^l \cos^2\frac{2\pi x}{l}dx - 2P \cdot A$$

Total potential energy:

$$V = \frac{4\pi^4 EJ}{l^3}A^2 - 2P \cdot A$$

The minimum condition for a function:

$$\frac{\partial V}{\partial A} = 0 \quad \Rightarrow \quad \frac{\partial V}{\partial A} = \frac{8\pi^4 EJ}{l^3}A - 2P = 0 \quad \Rightarrow \quad A = \frac{Pl^3}{4\pi^4 EJ}$$

An approximate function describing the deflection line:

$$\widetilde{w}(x) = \frac{Pl^3}{4\pi^4 EJ} \cdot \left(1 - \cos\frac{2\pi x}{l}\right)$$

Bending moment approximation:

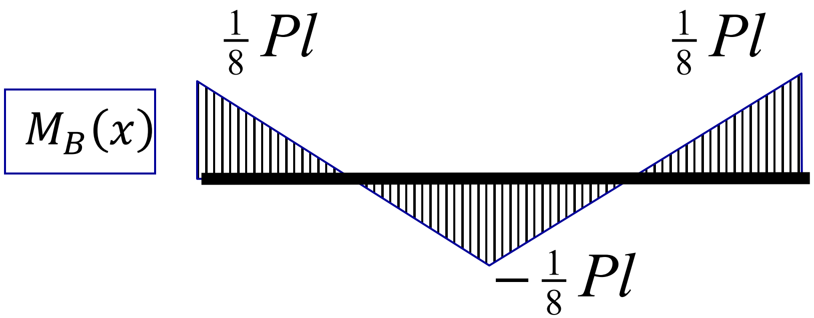
$$\widetilde{M}_B(x) = EJ \cdot \widetilde{w}''(x) = \frac{Pl}{\pi^2} \cdot \cos\frac{2\pi x}{l}$$

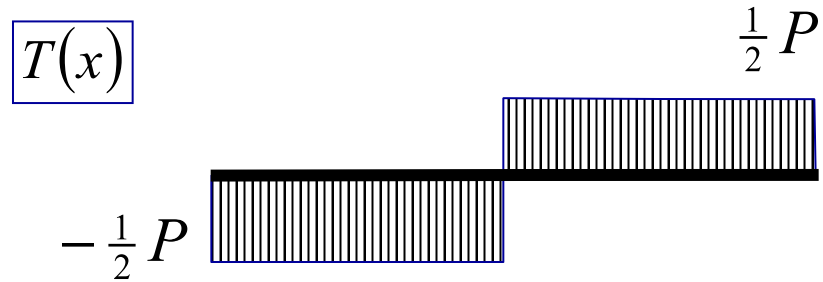
Shear force approximation:

$$\widetilde{T}(x) = EJ \cdot \widetilde{w}'''(x) = -\frac{P}{\pi} \cdot \sin\frac{2\pi x}{l}$$

Example 3: Beam fixed on both ends (cont.)

Exact solution:



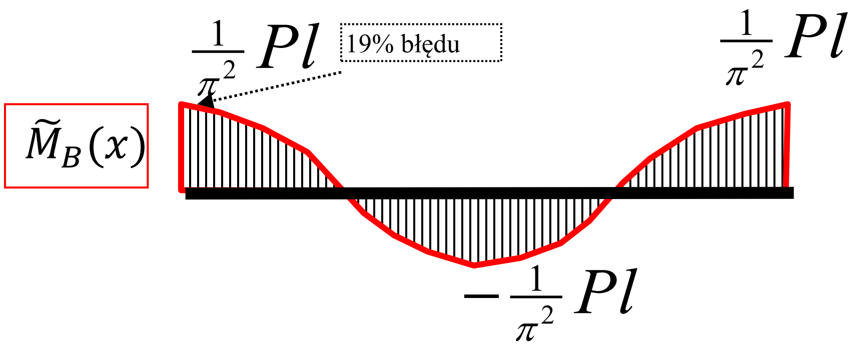


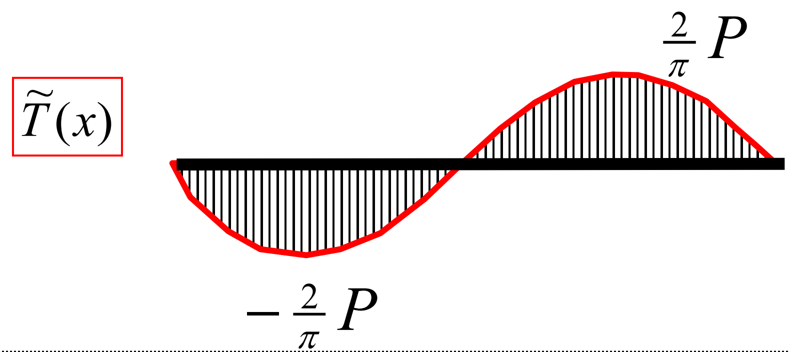
Approximate solution *(Ritz method)*:

$$\tilde{w}(x) = \frac{Pl^3}{4\pi^4 EJ} \cdot \left(1 - \cos\frac{2\pi x}{l}\right)$$

$$\tilde{M}_B(x) = \frac{Pl}{\pi^2} \cdot \cos\frac{2\pi x}{l}$$

$$\tilde{T}(x) = -\frac{P}{\pi} \cdot \sin\frac{2\pi x}{l}$$



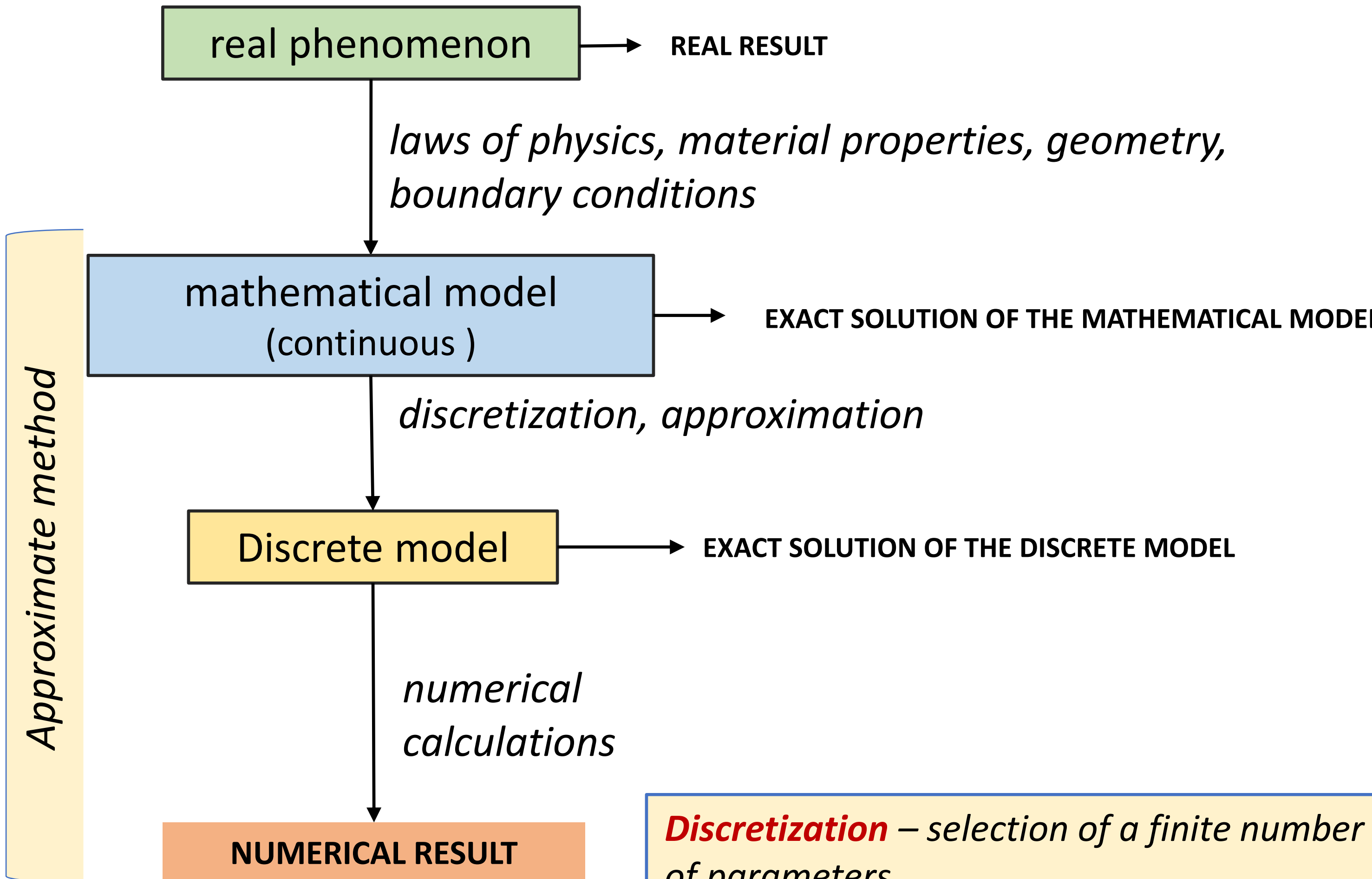


$$\tilde{w}_{extr} = \tilde{w}\left(\tfrac{1}{2}l\right) = \frac{-Pl^3}{2\pi^4 EJ} = -0.005133\,\frac{Pl^3}{EJ}$$

1.5% error

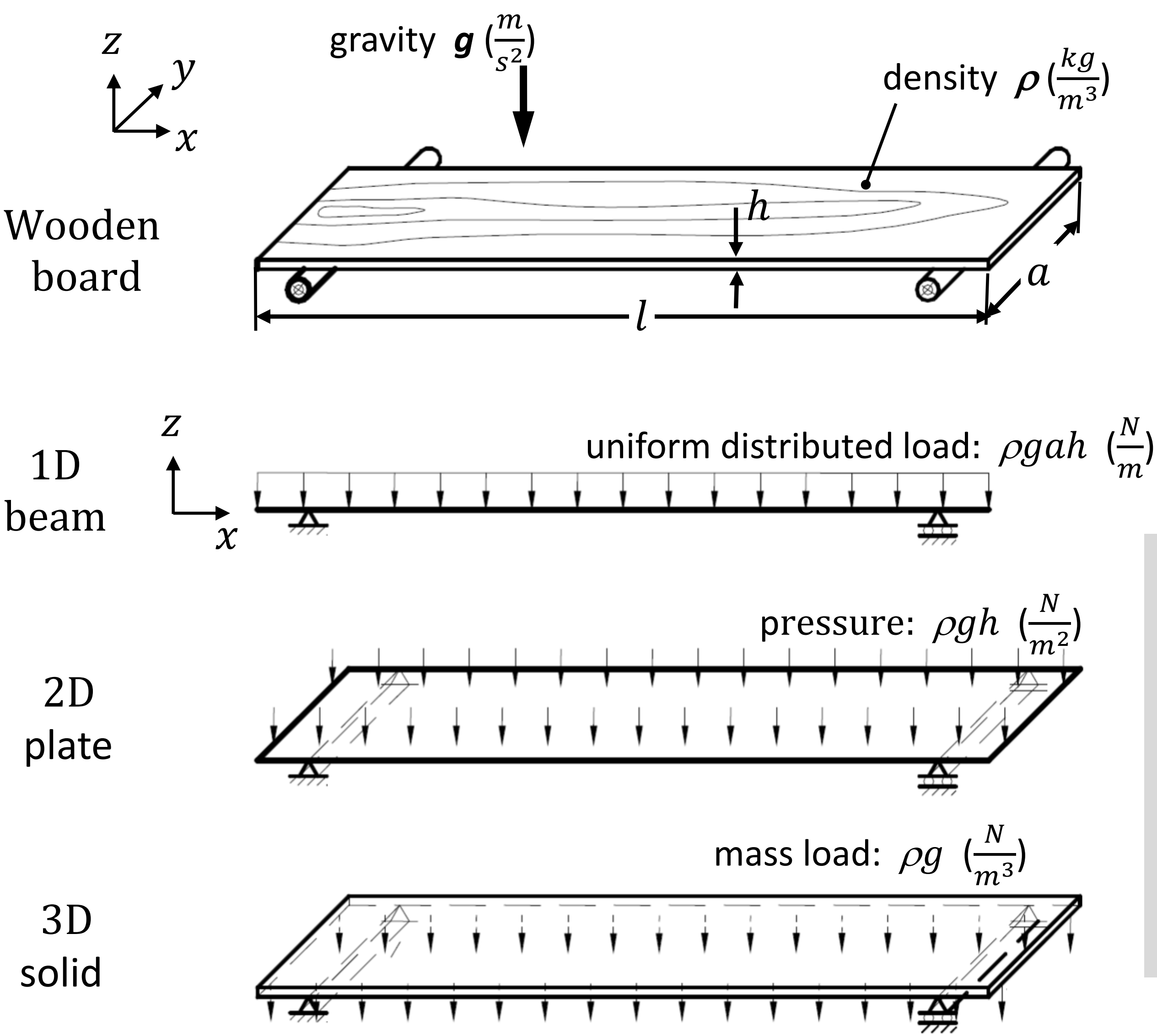
$$w_{extr} = w\left(\tfrac{1}{2}l\right) = \frac{-Pl^3}{192\,EJ} = -0.005208\,\frac{Pl^3}{EJ}$$

Approximate methods – flow chart



Discretization *– selection of a finite number of parameters,*
Approximation *– a method of description using pre-defined simple functions depending on the parameters being searched for*

Choosing a mathematical model



Assumptions:

Material properties:

- *isotropic,*
- *anisotropic,*
- *viscoelastic,*

Large deflections

Boundary conditions *(contact)*

There is no single best model!

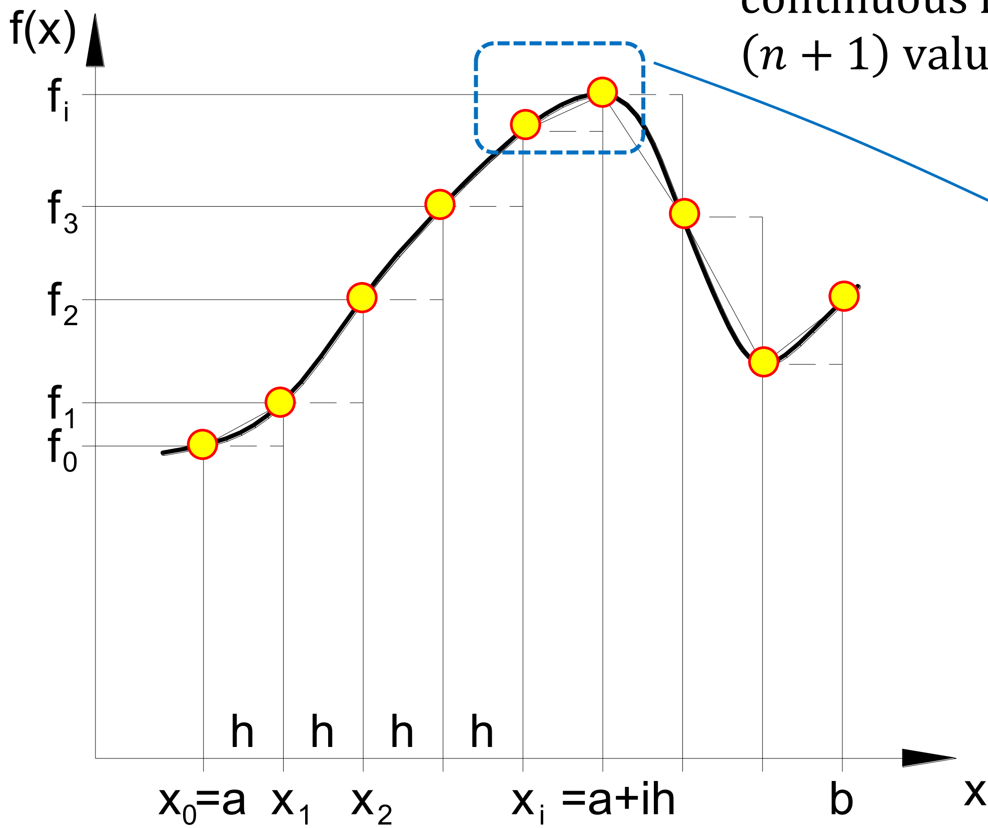
The right model depends on:

- *the purpose of the analysis,*
- *the design requirements,*
- *the desired accuracy of the results,*
- *the availability of material data,*
- *the available computational tools*

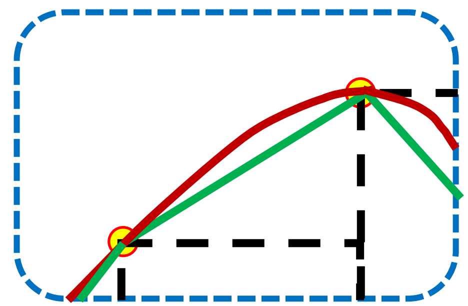
Discretization and approximation on the example of a function of one variable

Let′s consider the function: $\boldsymbol{f}(\boldsymbol{x})$ in the interval $< \mathrm{a}, \mathrm{b} >$

Let′s devide the interval $< \mathrm{a}, \mathrm{b} >$ into $\boldsymbol{n}$ equal subintervals of length: $h = (b - a)/n$



continuous function $\boldsymbol{f}(\boldsymbol{x})$ represents a set of $(n + 1)$ values: $\boldsymbol{f}(\boldsymbol{a} + \boldsymbol{ih}), i = 0, 1, 2, n$

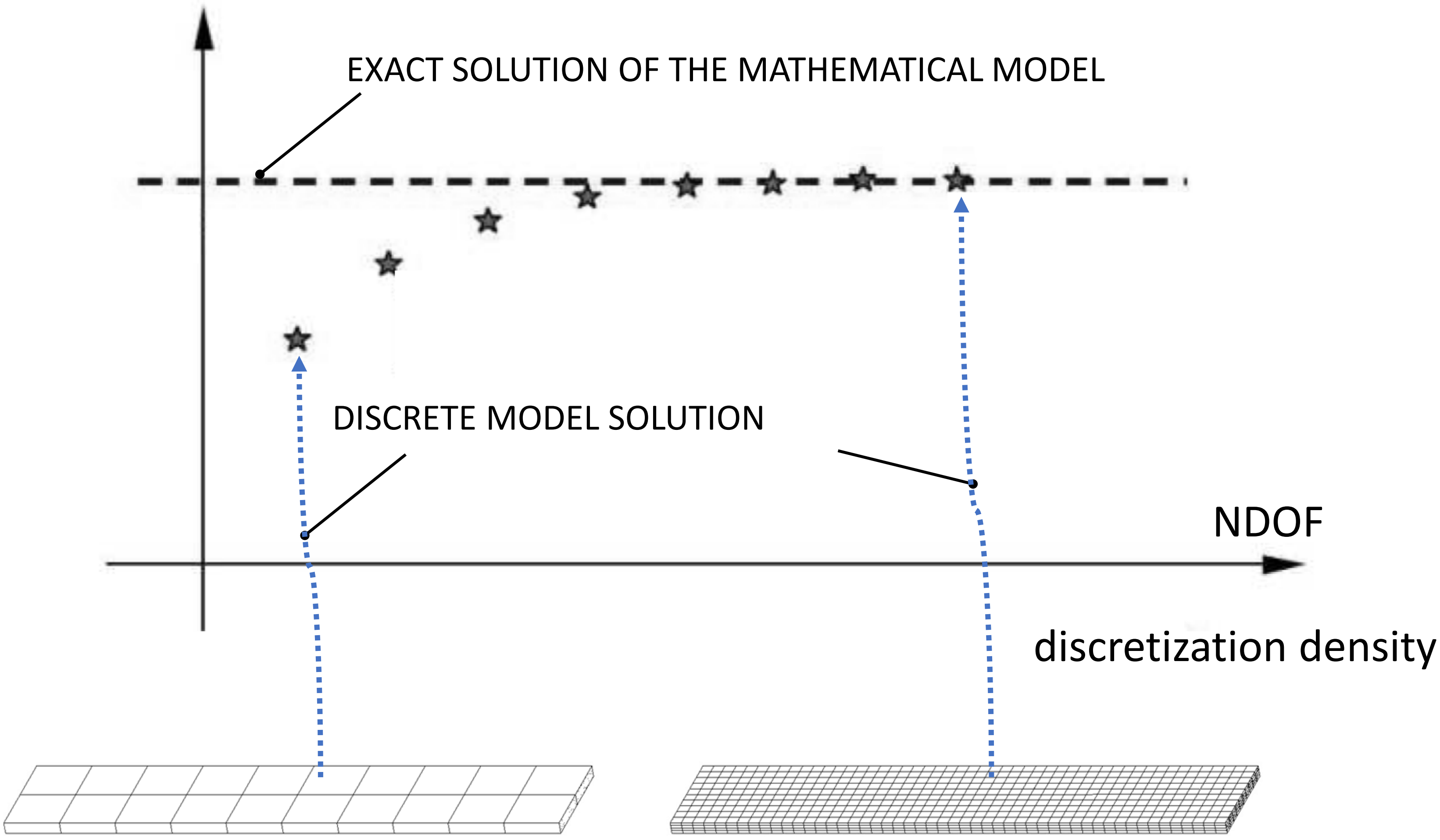


<u>The approximation can be:</u>

- *constant (stepped),*
- *linear,*
- *using splines.*

Discrete solution versus exact solution of the continuous problem

Discrete model



Approximate methods for analyzing the Poisson equation in 2D space

Poisson's equation is a partial differential equation with broad utility in electrostatics, mechanical engineering and theoretical physics:

- *Stationary heat flow,*
- *Stationary irrotational flow of an incompressible and inviscid fluid,*
- *Simple magnetic and electric fields,*
- *Stress distribution in the cross-section of a twisted rod*

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + f(x_1, x_2) = 0 ,$$

$$T(\overline{x}) = T_0 \quad , \quad \overline{x} \in \Gamma_u$$

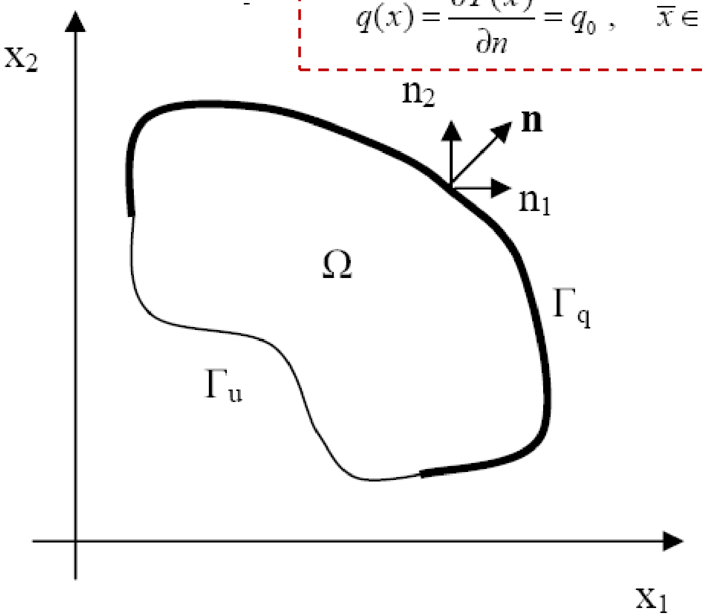
$$q(x) = \frac{\partial T(\overline{x})}{\partial n} = q_0 \ , \quad \overline{x} \in \Gamma_q$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x_1, x_2) = 0$$

Let's consider the boundary conditions:

$$u(\overline{x}) = u_0 \quad , \quad \overline{x} \in \Gamma_u$$ - Dirichlet on Γ_u

$$q(x) = \frac{\partial u(\overline{x})}{\partial n} = q_0 \ , \quad \overline{x} \in \Gamma_q$$ - Neumann on Γ_q



Where u_0 and q_0 are given functions defined on the corresponding parts of the boundary.
In special cases *(simple geometry and boundary conditions)* the task has its analytical solution.

Finite difference method (FDM)

FDM approximates the solution of a differential equation by replacing differential operators with **difference quotients**:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

then a reasonable approximation for that derivative would be to take

$$f'(a) \approx \frac{f(a+h)-f(a)}{h}$$

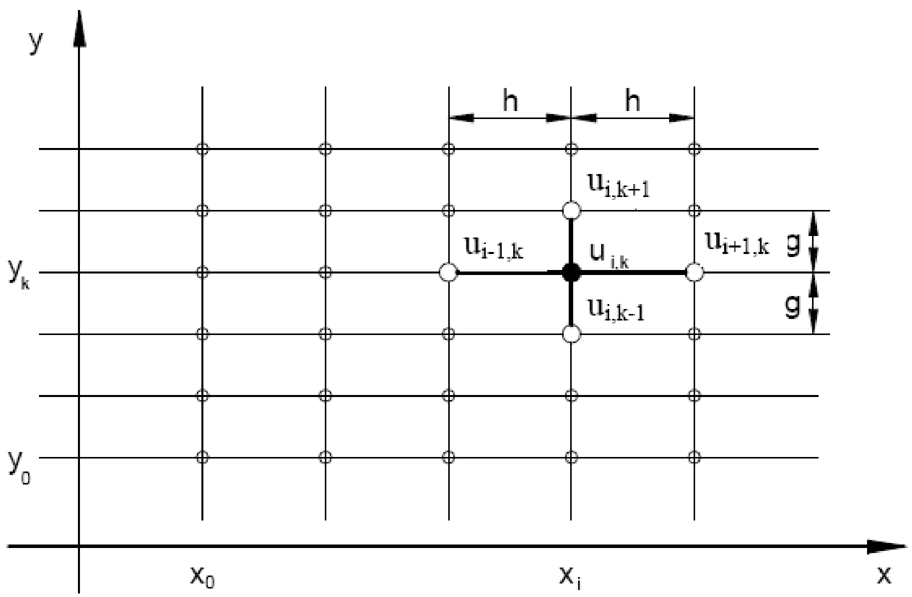
(difference quotient)

Discretization consists in replacing the searched function with a set of its values in the nodes of the grid *(regular or irregular).*

The derivative values at a point are replaced by the increments (differences) of the function at neighboring nodes.

Differential equations are replaced by algebraic equations, the so-called difference equations.

We obtain a system of algebraic equations with unknowns being the values of the function at the nodes.



For rectangular mesh:

$$x_i = x_o + ih, \quad y_k = y_o + kg, \qquad u_{i,k} = u(x_i, y_k)$$

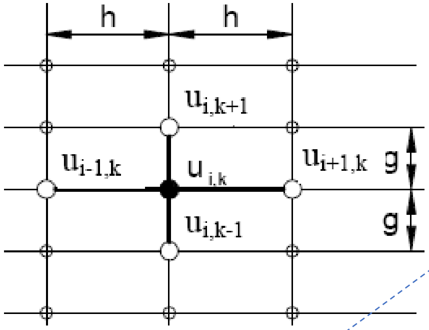
Different differential schemes can be adopted:

a) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k+1} - u_{i,k}}{g},$$ **Forward Differential quotient**

b) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k} - u_{i,k-1}}{g},$$ **Rear differential quotient**

c) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k+1} - u_{i,k-1}}{2g}.$$ **Central difference quotient**

Finite difference method (FDM)



Differences corresponding to higher derivatives:

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{\Delta^2 u}{\Delta x^2} = \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{h^2},$$

$$\frac{\partial^2 u}{\partial y^2} \simeq \frac{\Delta^2 u}{\Delta y^2} = \frac{u_{i,k+1} - 2u_{i,k} + u_{i,k-1}}{g^2}.$$

$$\frac{\partial^4 u}{\partial x^4} \simeq \frac{\Delta^4 u}{\Delta x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}$$

Using the difference diagram we can express the differential equation at any point (x_i, y_j) using an algebraic equation. In the case of the Poisson equation:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x_1, x_2) = 0 \quad \longrightarrow \quad \frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{1}{g^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) + f\left(x_i, y_j\right) = 0.$$

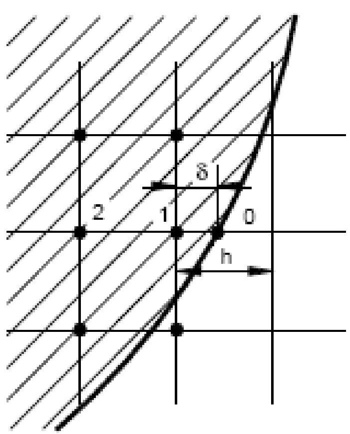
For regular grid *(h=g) and f=0 (Laplace's equation)* we get:

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}.$$

***N* grid points in the domain Ω, *N* equations, *N* unknowns** *(each equation corresponds to one grid node)*

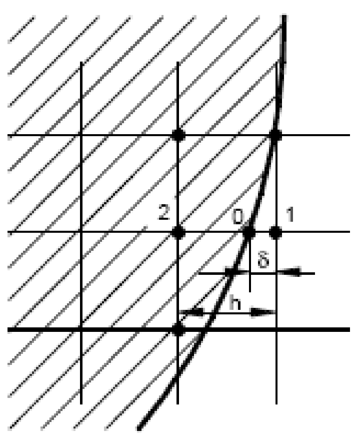
$$[A]\{u\} = \{b\}$$

a)



Boundary condition interpolation

b)



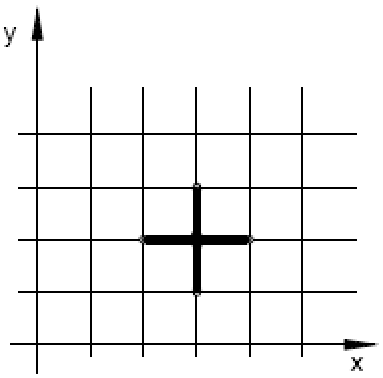
boundary condition extrapolation

In case of irregular boundary shape:

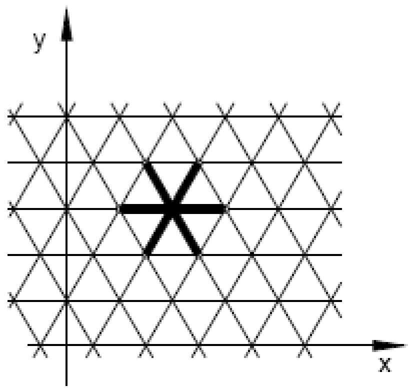
a) we assume $u_1 = \dfrac{hu_0 + \delta u_2}{h + \delta}$ instead of $u = u_0$

b) we assume $u_1 = \dfrac{hu_0 - \delta u_2}{h - \delta}$ instead of $u = u_0$

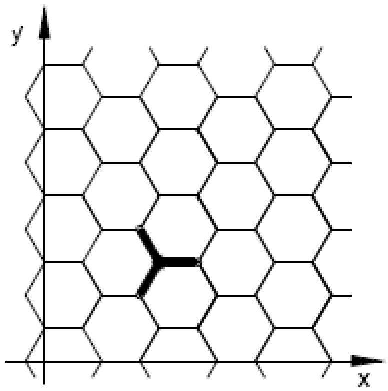
Examples of irregular grids in FDM



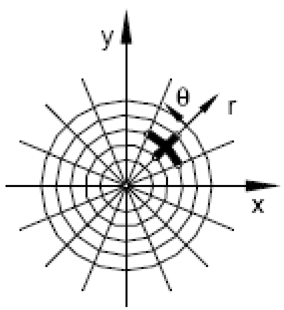
Square grid



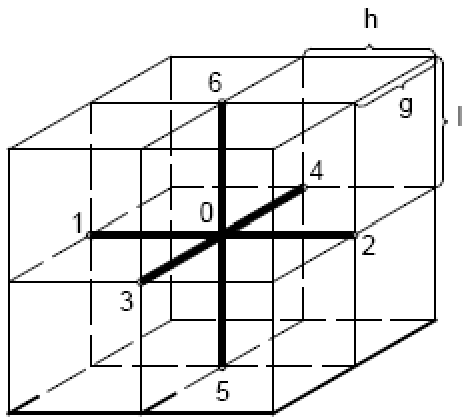
Triangular grid



Hexagonal grid



Polar grid



Rectangular grid

Boundary Element Method (BEM)

For the source point, one can present a boundary integral equation, which is an equivalent formulation to the Poisson problem:

$$c(\bar{\xi})u(\bar{\xi}) = \int_{\Gamma} u(x) q^{*}(\bar{\xi},\bar{x}) d\Gamma(x) - \int_{\Gamma} \frac{\partial u(\bar{x})}{\partial \bar{n}} u^{*}(\bar{\xi},\bar{x}) d\Gamma(\bar{x}) + \int_{\Omega} f(x) u^{*}(\bar{\xi},\bar{x}) dR(\bar{x})$$

Functions u* i q* depend on the position of two points: $\bar{\xi}$ - source point, $\bar{x}$ - observation point

$c(\bar{\xi})$ - the coefficient that takes the value of ½ on a smooth contour or 1 inside the area Ω

Kernel functions (known in the theory of integral equations):

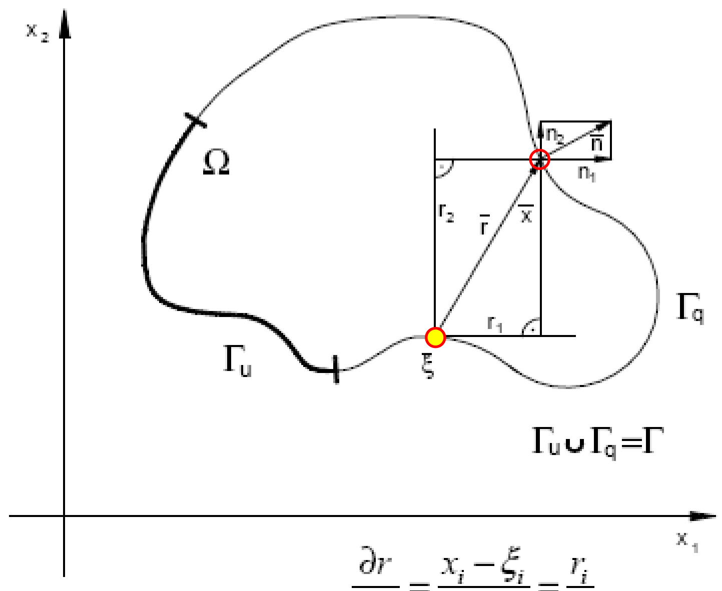
$$u^{*} = (\bar{\xi},\bar{x}) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right), \qquad r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}\,.$$

Function q* is defined by the directional derivative u*:

$$q^{*}(\bar{\xi},\bar{x}) = \frac{\partial u^{*}(\bar{\xi},\bar{x})}{\partial n}\,.$$

By differentiating, we get:

$$q^{*} = \frac{\partial u^{*}}{\partial x_1}\cdot n_1 + \frac{\partial u^{*}}{\partial x_2}\cdot n_2, \qquad q^{*} = \frac{-(r_1\cdot n_1 + r_2\cdot n_2)}{2\pi r^2},$$



$$\frac{\partial r}{\partial x_i} = \frac{x_i - \xi_i}{r} = \frac{r_i}{r}\,.$$

Where: $r_i = x_i - \xi_i$, i = 1,2, $\bar{n} = n_1, n_2$ is a unit external vector, normal to the boundary Γ

The boundary integral equation states the relation between $(\bar{x})$ and its derivative in normal direction $q(\bar{x}) = \frac{\partial u(\bar{x})}{\partial \bar{n}}$ on the boundary Γ.

Boundary Element Method (BEM) - the numerical approach

1. Discretization of the boundary (LE boundary elements)

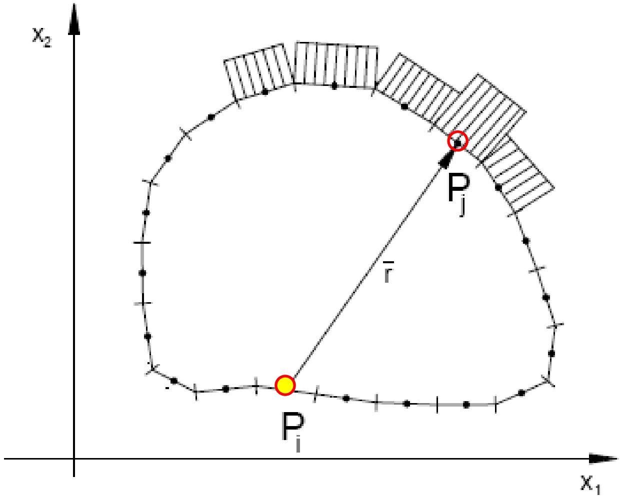
2. On each boundary element we approximate the functions $u(\bar{x})$ and $q(\bar{x})$

   (e.g.: $u(P_i)$ i $q(P_i)$ constant on boundary elements)

3. We transform the integral equation for each node $\bar{\xi}$ into an algebraic linear equation:

$$\frac{1}{2}u(P_i) = \sum_{j=1}^{LE}\int_{\Gamma_j} u^*(P_i,\bar{x})q(P_j)d\Gamma_j - \sum_{j=1}^{LE}\int_{\Gamma_j} q^*(P_i,\bar{x})u(P_j)d\Gamma_j + \int_{\Omega} f(\bar{x})u^*(P_i,\bar{x})dR$$

$$i = 1, 2, ..LE$$



after numerical integration for each grid point:

$$\frac{1}{2}u(P_i) = \sum_{j=1}^{LE} U_{ij}^* \cdot q(P_j) - \sum_{j=1}^{LE} Q_{ij}^* \cdot u(P_j) + f_i, \quad i = 1, 2...LE \,.\, f_i = \int_{\Omega} f(\bar{x})u^*(P_i,\bar{x})d\Omega(\bar{x})$$

LE linear equations with unknowns: $u(P_j)$ (if point $P_j \in \Gamma_q$) or $q(P_i)$ (if point $P_i \in \Gamma_u$)

$$\frac{1}{2}\{u\} = [U^*]\{q\} - [Q^*]\{u\} + \{f\}.$$

Finally: $$[A]\{y\} = \{b\}$$

The solution $\{y\}$ represents unknown boundary values of u and q. The matrix A – full, unsymmetric

4. Solution provides complete information about the function $u(\bar{x})$ and its derivative $q(\bar{x})$ on the boundary

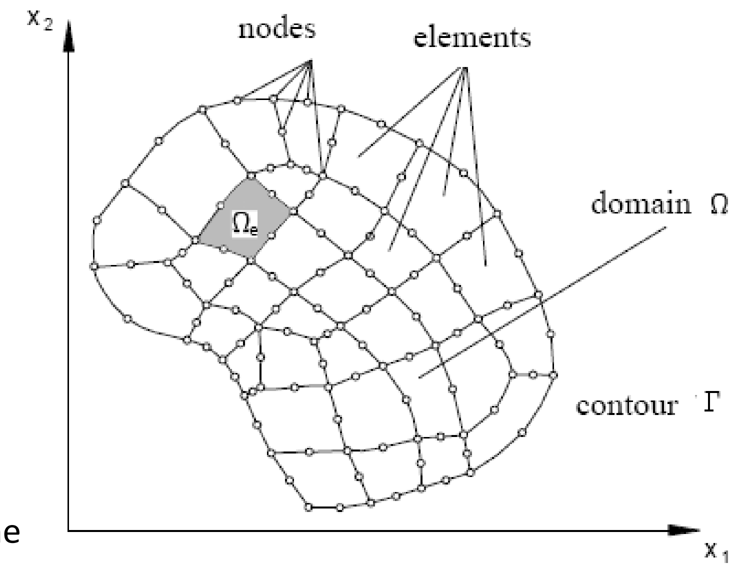
Finite Element Method (FEM)

Equivalent problem of minimizing of the fictional *(for the Poisson problem):*

$$I(u)=\frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega-\int_{\Gamma_q}q_0 u\,d\Gamma,$$

Where the function ***u*** satisfies the Dirichlet condition:

$$u(\bar{x})=u_0,\qquad \bar{x}\in\Gamma_u$$



1. Discretization of the solution domain Ω
   into elements Ωi , i= 1,LE connected in the nodes

$$\Omega=\bigcup_{i=1}^{LE}\Omega_e\qquad \text{i}\qquad \Omega_i\cap\Omega_j=0\qquad i\neq j,$$

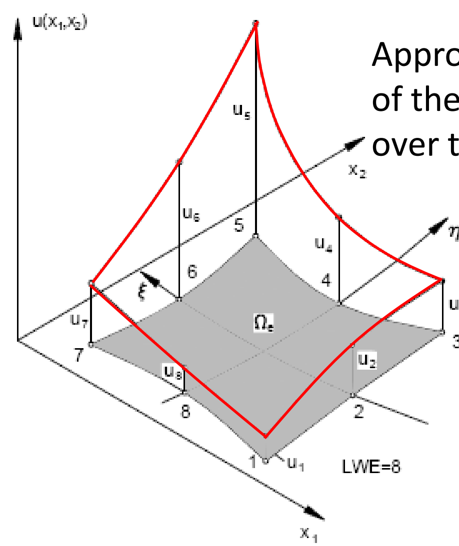
2. Approximation of function u$(\bar{x})$ within the finite element in the form of polynomials dependent on the unknown nodal values u_i

$$u(x_1,x_2)=\sum_{i=1}^{LWE}N_i(x_1,x_2)u_i$$

LWE – number of nodes of the element
u_i , i=1,….,LWE – nodal values of the approximated function
$N_i(x_1, x_2)$ – shape functions

3. Discrete form of the functional

$$I(u)\cong\sum_{i=1}^{LE}\frac{1}{2}\int_{\Omega_i}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega_i-\sum_{j=1}^{LK}\int_{\Gamma_j}q_0 u\,d\Gamma_j$$



Approximation
of the function $u(x,y)$
over the element Ω_e

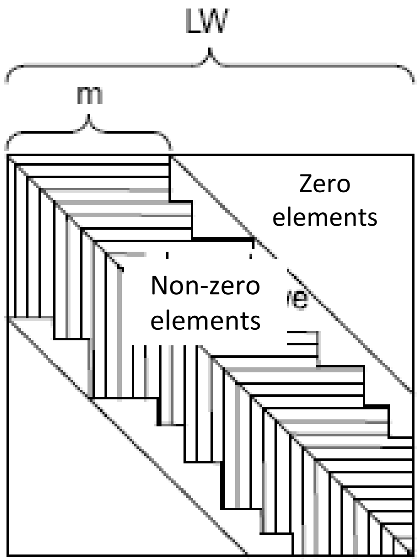
Finite Element Method (FEM)

Inside each element we have:

$$\frac{\partial u}{\partial x_1} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_1} u_i,$$
$$\frac{\partial u}{\partial x_2} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_2} u_i.$$

In this way the functional I is replaced by the function of several unknows u_i , i=1,2,….,LW , where LW enotes the number of nodes of the finite element mesh. In matrix form, this function has the form:

$$I(u) \approx \frac{1}{2}\lfloor u_1, u_2, u_3, \ldots, u_{LW} \rfloor \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1LW} \\ k_{21} & k_{22} & k_{23} & & \\ k_{31} & k_{32} & & & \\ \cdots & & & & \\ k_{LW1} & & & & k_{LW\,LW} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{LW} \end{Bmatrix} - \lfloor u_1, u_2, u_3, \ldots, u_{LW} \rfloor \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_{LW} \end{Bmatrix}$$



matrix: sparse, symmetrical, positive defined, banded

$$I \approx \frac{1}{2} \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times LW}{[K]} \underset{LW\times 1}{\{u\}} - \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times 1}{\{b\}}.$$

The necessary condition for the minimum of this function is that all partial derivatives are zero.:

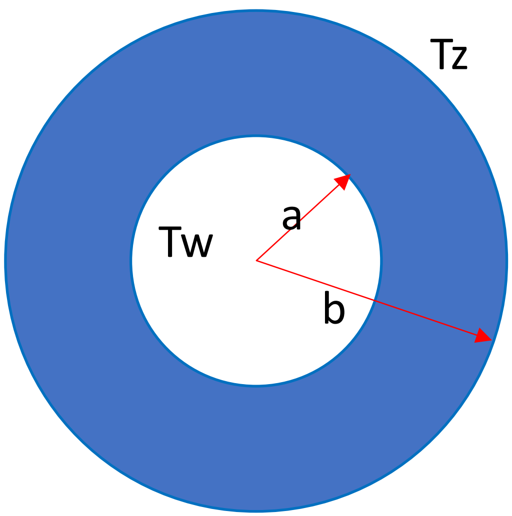
$$\frac{\partial I}{\partial u_i} = 0, \qquad i = 1, \ldots, LW.$$

$$[K]\{u\} = \{b\}, \text{ (+ Dirichlet b.c.)}$$

Set of the simultaneous equations with unknown nodal values of the investigated function

Example 4: Stationary heat flow in a pipe

The thick-walled steel pipe has an internal temp. T_w=100°C and the outside temp. T_z=20°C. Pipe inner radius is a=30mm, external radius b=40mm. Calculate the temperature distribution.
Data: λ=50W/mK.



Laplace's equation:

$$\nabla^2 T = 0$$

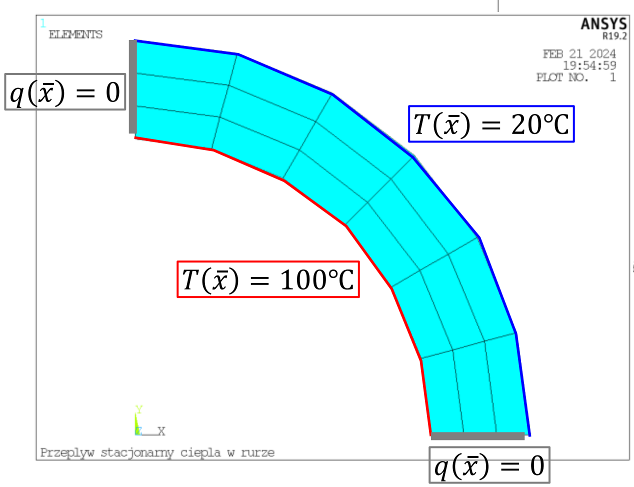
$$T(\bar{x}) = T_0 \quad , \quad \bar{x} \in \Gamma_u$$

$$q(x) = \frac{\partial T(\bar{x})}{\partial n} = q_0 \, , \quad \bar{x} \in \Gamma_q$$

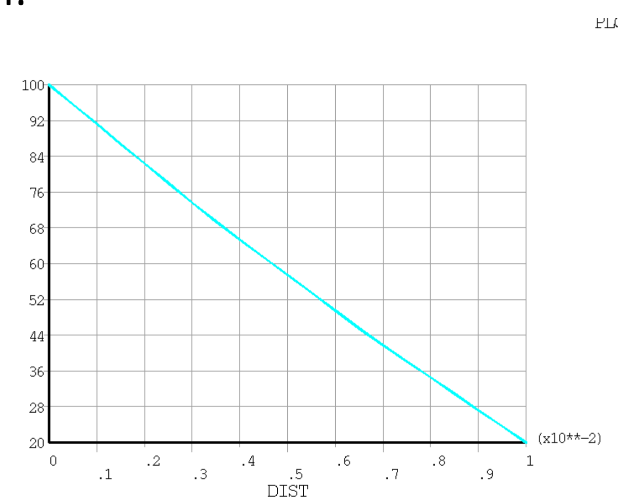
Analytical solution:

$$T(r) = T_w + \frac{T_z - T_w}{\ln\left(\frac{b}{a}\right)} \ln\left(\frac{r}{a}\right)$$

FEM model:



Temperature distribution:

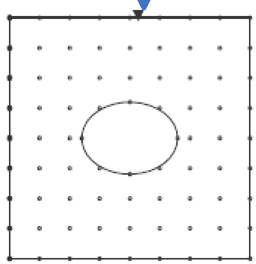


General calculation procedure

Finite difference method (FDM)

partial differential equations

Construction of a node grid and adoption of selected differential schemes



Replacing differential equations with difference equations for successive nodes of the region. Forming a system of linear equations.

Modification of the system of equations - introduction of boundary conditions

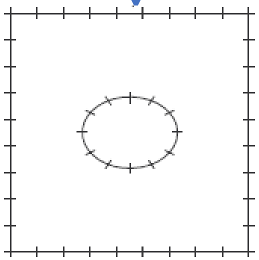
Solution of a system of linear equations (sparse, banded, usually symmetric matrix)

Complementary calculations, e.g. derivatives of the searched functions at the nodes

Boundary element method (BEM)

integral boundary equations

Division of the boundary into segments (boundary elements) and assumption of appropriate approximation functions on the elements (shape functions)



Construction of a discrete representation of the integral equation for successive boundary nodes. Formation of a system of linear equations

Modification of the system of equations - introduction of boundary conditions

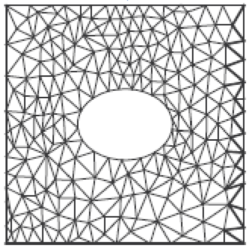
Solution of a system of linear equations (full asymmetric matrix)

Complementary calculations, e.g. derivatives of the searched in selected points of the domain

Finite element method (FEM)

functional minimization problem

Division of the region into small subregions (finite elements) and adoption of appropriate approximating functions on the elements (shape functions)



Construction of the stiffness matrix of successive elements. Formation of a system of linear equations.

Modification of the system of equations - introduction of boundary conditions

Solution of a system of linear equations (sparse, banded, usually symmetric matrix)

Complementary calculations, e.g. derivatives of the searched functions inside finite elements

FEM as an approximate method

Finite element method (**FEM**) is an approximate method that can be used as a numerical procedure to solve physical problems, including:

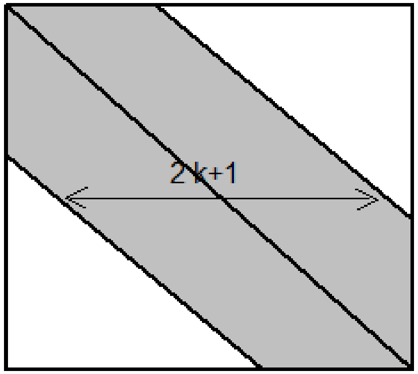
- solid mechanics,
- heat transfer,
- fluid flow,
- electromagnetism,
- coupled field problems
- ...

FEM was developed in the 1950s to solve problems for the civil and aerospace industries. The method has become the most powerful analytical tool, mainly due to the development of computers.

The purpose of the lecture: providing the basic knowledge and skills needed to understand and apply FEM to solve boundary value problems for partial differential equations.

Basics of matrix calculus

A banded matrix is a square matrix which nonzero elements lie on the main diagonal (diagonal) and on k lines parallel to the diagonal on each side ($a_{ij} = 0$ if $|i - j| > k$).

$[A] =$ 

The number $(2k + 1)$ is called the bandwidth and the number $(k + 1)$ is called the matrix $[A]$ half-band width.

A diagonal matrix with nonzero individual elements is called an identity matrix of dimension n where δ_{ik} denotes the Kronecker symbol:

$\delta_{ik} = 1$ if $i = k$, $\delta_{ik} = 0$, if $i \neq k$.

$$[I] = [\delta_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Transposed matrix of marix $\underset{m\times n}{[A]} = [a_{ik}]$ is the matrix $\underset{n\times m}{[A]^T} = [a_{ki}]$ created by interchanging the rows and columns. We have: $\lfloor q \rfloor^T = \{q\}$ i $\{q\}^T = \lfloor q \rfloor$.

A square matrix is called a symmetric matrix if: $[A]^T = [A]$ $(a_{ik} = a_{ki})$.

Basic operations on matrices

The sum of matrices: $\underset{m\times n}{[A]} = [a_{ik}]$ and $\underset{m\times n}{[B]} = [b_{ik}]$ is matrix $\underset{m\times n}{[C]} = [a_{ik} + b_{ik}]$.

The matrix addition operation requires that the dimensions of the component matrices be consistent.

The product of matrix $\underset{m\times n}{[A]} = [a_{ik}]$ by a real number λ gives matrix $\underset{m\times n}{[B]} = [\lambda a_{ik}]$.

The product of a matrix $\underset{m\times n}{[A]} = [a_{ik}]$ by a matrix $\underset{n\times p}{[B]} = [b_{ik}]$ gives matrix $\underset{m\times p}{[C]} = [c_{ik}]$

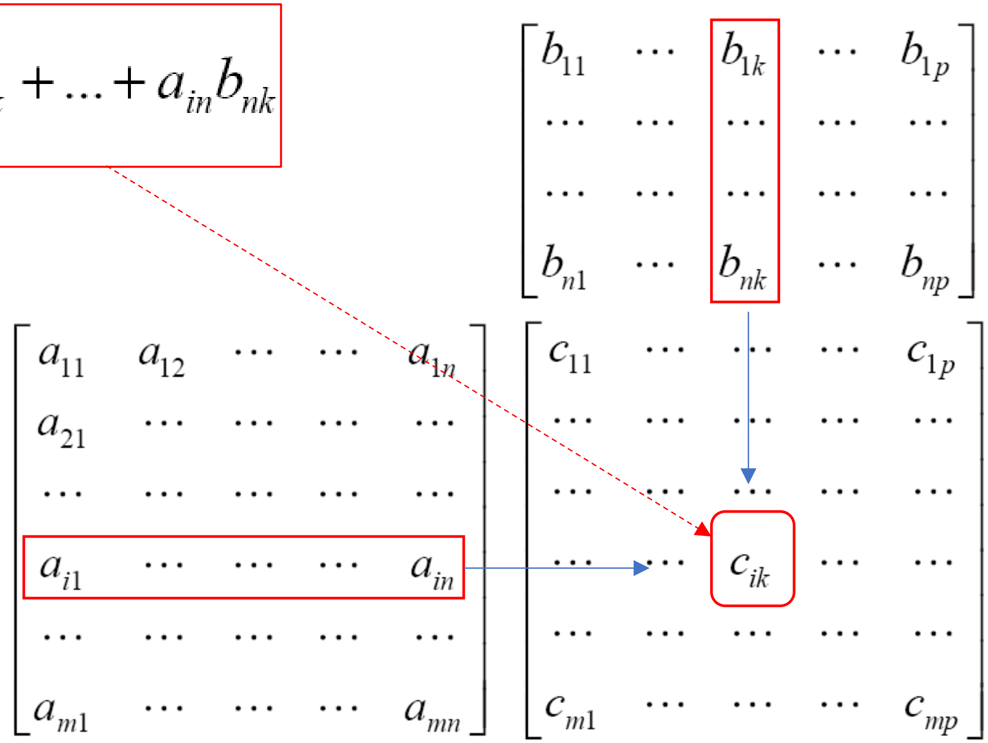
such that:

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} \qquad i = 1,2,\ldots,m, \quad k = 1,2,\ldots,p. \qquad \text{(D.3)}$$

Matrix multiplication is only possible if the number of columns of the first matrix is equal to the number of rows of the second matrix. Matrix multiplication is not commutative $([A][B] \neq [B][A])$.

Matrix multiplication using Falk's scheme

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk}$$



Selected properties of matrix operations:

1. $[A]\cdot([B]\cdot[C]) = ([A]\cdot[B])\cdot[C]$
2. $\alpha[A]\cdot[B] = (\alpha[A])\cdot[B] = [A]\cdot(\alpha[B])$
3. $([A]\cdot[B])^T = [B]^T[A]^T$

Determinant of a matrix

The determinant of a square matrix $\underset{n\times n}{[A]} = [a_{ik}]$ is a real number $det[A]$ which is defined by the relations

1) for $n = 1$ $det\underset{1\times 1}{[A]} = a_{11}$
2) for $n = 2$ $det\underset{2\times 2}{[A]} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$ (D.7)
3) for $n \geq 3$ the determinant can be calculated by selecting any row r and using the so-called Laplace expansion:

$$det[A] = a_{r1}\alpha_{r1} + a_{r2}\alpha_{r2} + \cdots + a_{rn}\alpha_{rn} = \textstyle\sum_j a_{rj}\,\alpha_{rj} \quad \text{(D.8)}$$

α_{rj} is called the algebraic complement of element a_{rj} of matrix [A] and is calculated according to the formula

$$\alpha_{rj} = (-1)^{r+j}\det\left[M_{rj}\right] \quad \text{(D.9)}$$

where $[M_{rj}]$ is a submatrix of matrix [A] obtained by deleting the rth row and jth column of matrix [A].

In particular, for $n = 3$ we get by choosing $r = 1$:

$$\det[A] = a_{11}\cdot(a_{22}\cdot a_{33} - a_{23}\cdot a_{32}) - a_{12}\cdot(a_{21}\cdot a_{33} - a_{23}\cdot a_{31}) + a_{13}\cdot(a_{21}\cdot a_{32} - a_{22}\cdot a_{31})$$

A matrix whose determinant is equal to zero is called a singular matrix.

The rank of the matrix A is the largest dimension of the square submatrix created by deleting part of the rows and columns for which the determinant is different from zero. The rank of a nonsingular matrix of dimension n is therefore n. The rank of a singular matrix is smaller than its dimension.

Selected properties of the determinant:

1. If any two rows (columns) are linearly dependent (can be represented as a linear combination of the others), then the value of the determinant is equal to zero.
2. $det[A] = det[A]^T$.
3. The determinant of a diagonal matrix is equal to the product of its diagonal elements.
4. $det([A] \cdot [B]) = det[A] \cdot det[B]$.

The inverse matrix of a nonsingular matrix $\underset{n\times n}{[A]}$ is called a matrix $[A]^{-1}$ such that:

$$[A]\cdot[A]^{-1}=[A]^{-1}[A]=[I]=[\delta_{ik}].$$

There is exactly one inverse matrix of a nonsingular matrix

$$[A]^{-1}=\frac{1}{\det A}[\alpha_{ik}]^{T},$$

where α_{ik} are the algebraic complements of the elements a_{ik} of the matrix [A].

<u>A system of m linear equations with n unknowns</u>

$$\sum_{j=1}^{n}a_{ij}\,x_j=b_i\quad(i=1,2,\cdots,m)$$

can be written in matrix form

$$\underset{m\times n}{[A]}\,\underset{n\times 1}{\{x\}}=\underset{m\times 1}{\{b\}}$$

A system is called inconsistent when it has no solution, determinate when it has exactly one solution, or indeterminate when it has infinitely many solutions.

Examples

Example 1

Let $[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Let's calculate: $[A] \cdot [A]^T$.

According to Falk's scheme:

$$\begin{array}{cc} & \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} & \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \\ 17 & 39 & 61 \end{bmatrix} \end{array}$$

Finally: $[C] = [A] \cdot [A]^T = \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \\ 17 & 39 & 61 \end{bmatrix}$.

Example 2

Prove that the stiffness matrix of the beam element [k] is singular

$$[k] = \frac{2EJ}{l^3} \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

In the matrix [k] we can see a linear relationship between the rows (and columns), hence the conclusion that the determinant must be zero. For example, the first row is the result of multiplying the third row by –1. The first row can also be obtained by summing the second and fourth rows and dividing the result by *l*.

Examples

EXAMPLE 3

Let $[A] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let's calculate: $[A]^{-1}$.

First, we determine the value of the determinant:

$$det[A] = 2 \cdot (-1)^2 \cdot (1 \cdot 1 - 0 \cdot 0) + 1 \cdot (-1)^3 \cdot (1 \cdot 1 - 0 \cdot 0) + 1 \cdot (-1)^4 \cdot (1 \cdot 0 - 1 \cdot 0) = 2 - 1 = 1$$

$det[A] = 1$.

This means that there is an inverse matrix.

The matrix of algebraic complements is in this case equal to:

$$[\alpha_{ik}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Hence $[A]^{-1} = \frac{1}{det[A]}[\alpha_{ik}]^T$ => $[A]^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Verification:

$$[A] \cdot [A]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I].$$